Roots of Polynomials

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1 Introduction

For most polynomial problems, determining information on their roots is important. Common things include: the existence of rational roots, number of real roots, signs of real roots, sizes of roots, etc. We will consider some techniques useful in this endeavor, going through Descartes' Rule of Signs more closely.

2 Recap of Some Common Results

Theorem 2.1 (Rational Roots Test). *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients. Then if $\frac{a}{b}$ is a rational root of $f(x)$ with $\gcd(a,b) = 1$, then $a \mid a_0$ and $b \mid a_n$.*

Not only does the rational roots test provide a good way to generate some roots, it also provides a way to test for any linear integer polynomial factors of $f(x)$. This can be key in determining irreducibility, and if the degree of $f(x)$ is 2 or 3 then checking that $f(x)$ has no rational roots is sufficient.

Theorem 2.2 (Vieta's Formulas). *Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial with roots $r_1, r_2, \ldots, r_n$. Then for all $1 \le j \le n$ we have*

$$\sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} r_{i_1} r_{i_2} \cdots r_{i_j} = (-1)^j a_j$$

Using Vieta's formulas one can get expressions for any symmetric function in the $r_1, \ldots, r_n$ in terms of the coefficients $a_1, \ldots, a_n$. We can also get some bounds for certain roots: for example, there exists a root r for which $|r| \ge \frac{|a_1|}{n}$, one with $|r| \ge \sqrt[n]{|a_n|}$ (similarly with $\le$).

Theorem 2.3 (Intermediate Value Theorem). *If $f(x)$ is a continuous function (for example, a polynomial) sending real numbers to real numbers, and a, b are real numbers with $f(x) < 0 < f(b)$, then there exists an x between a, b such that $f(x) = 0$.*

The intermediate value theorem is useful in determining the existence of a root of the polynomial, as well as a bound on its size.

Theorem 2.4 (Lagrange Interpolation). *Let $x_1, x_2, \ldots, x_n$ be n distinct real numbers ($n \ge 1$), and $y_1, y_2, \ldots, y_n$ arbitrary real numbers. Then there exists a polynomial of degree at most $n-1$ for which $y_i = f(x_i)$ for all $1 \le i \le n$. Furthermore, a possible choice of $f(x)$ is*

$$f(x) = \sum_{i=1}^{n} y_i \prod_{j=1, j \ne i}^{n} \frac{x - x_j}{x_i - x_j}$$

3 Example Problem

Problem 3.1. *Show that for any monic polynomial $f(x)$ of degree n with real coefficients is the average of two monic polynomials $g(x), h(x)$ of degree n, each with n real roots. (USAMO 2002)*

Proof. The key idea is to figure out how to construct polynomials having all real roots, while controlling their behaviour (in this case, having two such which average to $f(x)$. One great way is to use Lagrange interpolation, ensuring the n real roots each via intermediate value theorem. To make g, h average to f, we need to make sure $(g+h)/2$ agrees with f in enough places. Indeed, polynomials of degree n are equal if they agree in $n+1$ places. Furthermore, monic polynomials of degree n are equal if they agree at n distinct points. With this in mind, define a_i, b_i for $1 \leq i \leq n$ by

$$\begin{aligned} a_i + b_i =& 2f(i) \\ (-1)^i a_i >& 0 \\ (-1)^i b_i <& 0 \end{aligned}$$

As one of a_i, b_i is positive and one negative, it is easy to see that this is possible. If polynomials g, h satisfy $g(i) = a_i$ and $h(i) = b_i$, then by intermediate value theorem, they have at least $n-1$ real roots. By Lagrange interpolation, there exist polynomials g, h of degrees at most $n-1$ satisfying this. However, we note that we don't have monic polynomials of degree n! To fix this, consider also stipulating that

$$g(0) = c_1, \qquad h(0) = c_2.$$

The resulting Lagrange interpolated polynomials have degree $\leq n$ and leading term being a non-constant linear function in c_1, c_2, so we can choose c_1, c_2 such that they are both 1. Of course, as before these polynomials have at least $n-1$ roots, hence n since the degree of the polynomials is n. □

4 Descartes' Rule of Signs

Descartes' Rule of Signs is a technique which can be useful when considering roots of polynomials. While it may not be the most common trick which appears in polynomial problems, it can be powerful when applicable. The statement is as follows:

Theorem 4.1 (Descartes' Rule of Signs)**.** *Let $f(x)$ be a polynomial with real coefficients, and we order the terms be decreasing exponent, and let there be S sign differences between consecutive non-zero coefficients. Then the number of positive real roots is at most S and has the same parity as S.*

More explicitly, let $f(x) = a_n x^{e_n} + a_{n-1} x^{e_n - 1} + \cdots + a_1 x^{e_1}$, where $e_n > e_{n-1} > \cdots > e_1 \geq 0$ are integers, and $a_n, a_{n-1}, \ldots, a_1$ are non-zero real numbers. Then consider the sequence $a_n, a_{n-1}, \ldots, a_1$; S is the number of sign changes in this sequence. It can be proven by induction, and is a good exercise.

Some uses/things to note about Descartes' Rule are:

- Applying it to $f(-x)$, you can determine the same information about the negative roots (a bound on how many and their parity)
- Proving there exists a real root: if S is odd, then the number of positive roots is odd, hence non-zero
- If $S = 1$, then there is exactly one positive real root
- By applying it to $f(x)$ and $f(-x)$ and using the bounds, you may be able to prove that there are roots which are not real (don't forget to check if 0 is a root as well)
- If you know how many positive/negative roots $f(x)$ has, you can deduce things about the signs of the coefficients

For example, the polynomial $f(x) = x^n + x + 1$ gives the sequence $[1, 1, 1]$, hence no sign changes and no positive roots. We get $f(-x) = (-1)^n x^n - x + 1$, so $[(-1)^n, -1, 1]$. Thus if n is even there are 2 sign changes, and if n is odd there is one sign change. To summarize, we get

- If n is even, then f has 0 or 2 negative real roots and 0 positive roots.
- If n is odd, then f has 1 negative real root and 0 positive roots.

Thus if $n \geq 3$ ($n = 2$ happens to work as well), we see that $f(x)$ does not have all real roots.

5 Using Descartes' Rule of Signs

Problem 5.1. *Prove that there does not exist a multiset S (i.e. a collection like a set but we allow repetition) of 2000 not necessarily distinct non-zero real numbers, such that for any multiset of 1000 of these numbers, there exists a polynomial having these numbers as its roots, with lead coefficient 1 and the other coefficients being a permutation of the remaining 1000 numbers?*

Proof. Let there be p positive elements and n negative elements of the multiset, so that $p+n=2000$. Hence one of p,n is at least 1000, so assume first that $p \geq 1000$. Take S_1 to be any multiset of 1000 of the positive elements, and take this to be our multiset of roots. From Descartes' rule of signs, we must have all 1000 possible sign changes, whence of the remaining numbers, 500 are positive and 500 negative, so $p=1500$. Now choose S_1 to be all 500 negative roots and any 500 positive ones. The corresponding $f(x)$ must have at least 500 sign changes, but its coefficients are all positive, contradiction.
For the case $n \geq 1000$, let S_2 be a set of 1000 negative elements, and $S_1 = S - S_2$. The $f(x)$ corresponding to S_1 has leading coefficient 1 and the rest negative, i.e. one sign change, and hence exactly one positive root. Therefore $p=1$ and $n=1999$, so take S_1 to be any 1000 negative roots. The corresponding $f(x)$ must have an even number of sign changes, and it has 2 positive coefficients (remembering the leading coefficient) and 999 negative ones. The only way this is possible is if the other positive coefficient is the constant one. But then the signs of $f(-x)$ are $[1,1,-1,1,-1,1,\dots,-1,1,-1,1,1]$, which is only 998 sign changes, contradicting $f(x)$ having 1000 negative roots. Therefore, such a multiset does not exist.

$\square$

6 Problems

1. The roots of the equation

$$x^3 - 3ax^2 + bx + 18c = 0$$

form a non-constant arithmetic progression, and the roots of

$$x^3 + bx^2 + x - c^3 = 0$$

form a non-constant geometric progression. Given that a, b, c are all real numbers, find all possible positive integral values of a and b.

2. Let $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = 0$ be a polynomial with real coefficients such that $0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq 1$. If λ is a root with $|\lambda| \geq 1$ then $\lambda^{n+1} = 1$.

3. Let $P(x)$ be a polynomial of degree 2012 with real coefficients satisfying:

$$P(a)^3 + P(b)^3 + P(c)^3 \geq 3P(a)P(b)P(c)$$

for all reals a, b, c such that $a+b+c=0$. Is it possible for $P(x)$ to have exactly 2012 distinct real roots?

4. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a nonzero polynomial with integer coefficients such that $P(r) = P(s) = 0$ for some integers r, s with $0 < r < s$. Prove that $a_k \leq -s$ for some k.

5. [some elementary calculus may be required] Let $a_1, a_2, \ldots, a_m$ be non-zero real numbers satisfying

$$1^k \cdot a_1 + 2^k \cdot a_2 + \cdots + m^k \cdot a_m = 0$$

for $k = 0, 1, \ldots, n$. Prove that the sequence $a_1, a_2, \ldots, a_m$ changes sign at least $n+1$ times.

6. Let n be a positive integer which is at least 2. David and Jacob play a game with a polynomial of degree $2n$,

$$x^{2n} + \Box x^{2n-1} + \Box x^{2n-2} + \cdots + \Box x + 1$$

where they fill in real numbers into empty boxes in turn. David wins if the resulting polynomial has no real root, and Jacob wins if it does.
a) If Jacob goes first, who wins?
b) If David goes first, who wins?

7. Let $P(x) = x^2 + 2007x + 1$. Prove that for every positive integer n, the equation $P(P(\cdots P(x))\cdots) = 0$ has at least one real solution, where the composition is performed n times.

8. Assume the polynomials $P(x), Q(x), R(x)$ have real coefficients and satisfy $P(x)^2 + Q(x)^2 = R(x)^2$. If the degrees of P, Q, R are $2, 3, 3$ respectively, prove that at least one of $Q(x), R(x)$ has 3 real roots.

9. Does there exist a finite set S of non-zero real numbers such that for any positive integer n there exists a polynomial $P(x)$ with degree at least n, such that all the roots and coefficients are from S?

10. Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

11. Find all real coefficient polynomials $f(x)$ which satisfy the following: i) $f(x) = a_0x^{2n} + a_2x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n}, a_0 > 0$
ii) $\sum_{j=0}^{n} a_{2j}a_{2n-2j} \leq \binom{2n}{n} a_0a_{2n}$
iii) All the roots of $f(x)$ are imaginary numbers with no real part.

12. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of n distinct numbers, for some $n \geq 9$, exactly $n-3$ of which are real. Prove that there are at most two quadratic polynomials $f(x)$ with complex coefficients for which $f(S) = S$ (i.e. f permutes the elements of S).